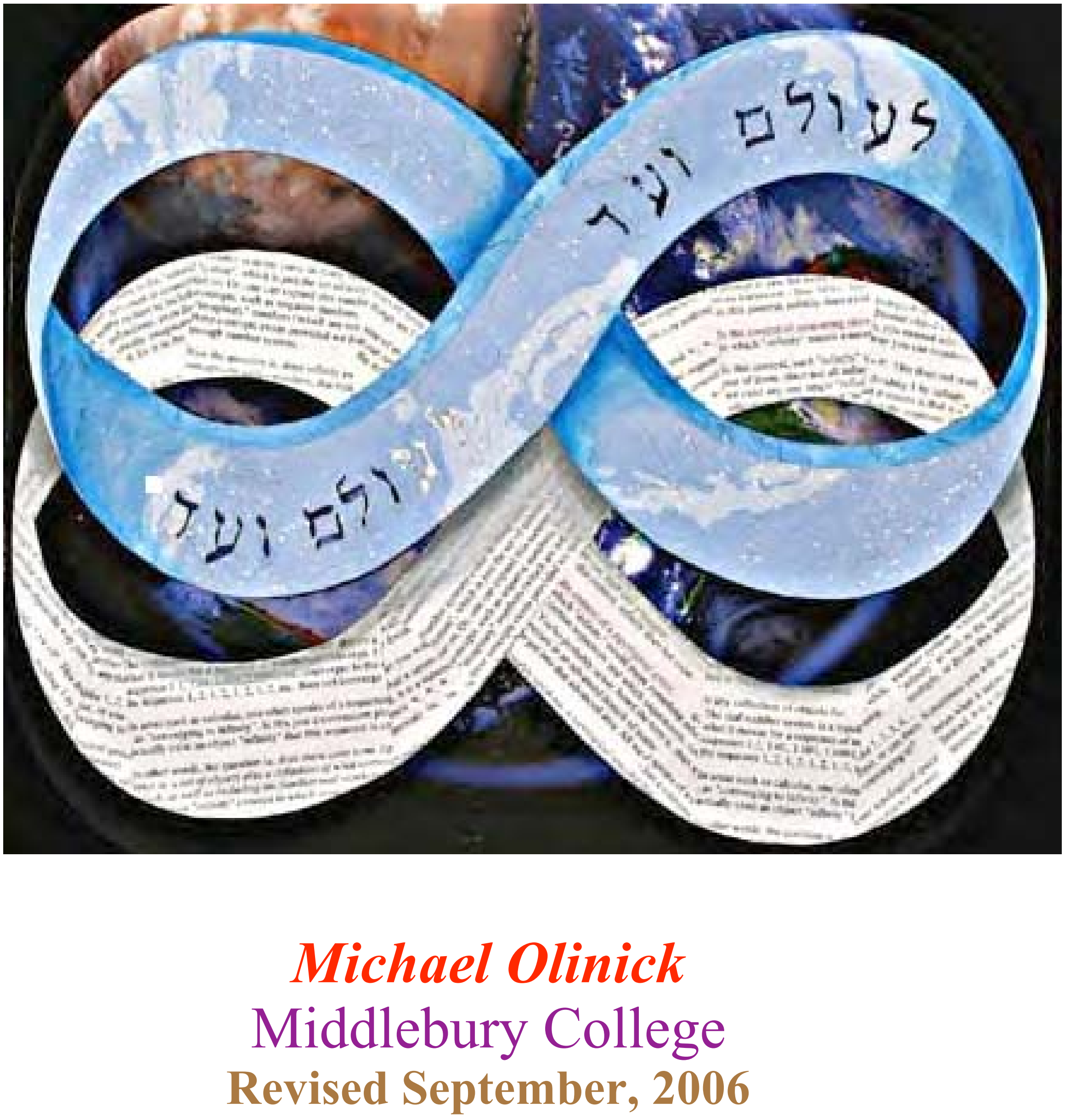
Discovering Infinity

Some Mathematical Notes for FYSE 1176



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Revised September, 2006

The image on the front page is a collage “Infinity” by the contemporary American artist Anna Fine Foer (http://annafineart.com). It is a multimedia collage on archival-quality paper and measures 14” by 20”

Foer notes:

> This collage piece explores spiritual and mathematical expressions of infinity. The Hebrew words לעולם ועד l'olam va'ed" ("forever and ever") are part of the Shema, the most basic prayer in Judaism; the globe in the background echoes the use of "olam," which can mean "world," "universe," or "infinity." The Möbius strip is a mathematical symbol for infinity, while the text represents mathematical proofs for the determining the existence of infinity.

I. What Is It All About?

From *Webster's New Collegiate Dictionary:*

Main Entry: 1 ***in·fi·nite***
Pronunciation: 'in-f&-n&t
Function: adjective
Etymology: Middle English infinit, from Anglo-French or Latin; Anglo-French, from Latin infinitus, from in- + finitus finite
1 : extending indefinitely : ENDLESS <infinite space>
2 : immeasurably or inconceivably great or extensive : INEXHAUSTIBLE <infinite patience>
3 : subject to no limitation or external determination
4 a : extending beyond, lying beyond, or being greater than any preassigned finite value however large <infinite number of positive numbers> b : extending to infinity <infinite plane surface> c : characterized by an infinite number of elements or terms <an infinite set> <an infinite series>

Main Entry***: in·fin·i·ty***
Pronunciation: in-'fi-n&-tE
Function: noun
Inflected Form(s): plural -ties
1 a : the quality of being infinite b : unlimited extent of time, space, or quantity : BOUNDLESSNESS
2 : an indefinitely great number or amount <an infinity of stars>
3 a : the limit of the value of a function or variable when it tends to become numerically larger than any preassigned finite number b : a part of a geometric magnitude that lies beyond any part whose distance from a given reference position is finite <do parallel lines ever meet if they extend to infinity> c : a transfinite number (as aleph-null)
4 : a distance so great that the rays of light from a point source at that distance may be regarded as parallel

II. The Heart of the Matter

Mathematicians today regard the concept of **set** to be foundation of all of mathematics and thus the key underlying everything we know about Infinity. Since it is impossible to define all words, we will assume that we know what it means for something to be a set and what it means for something to be an element of a set. The words "set" and "collection" will be used synonymously. As Paul Halmos writes in his elegant book *Naïve Set Theory,* "A pack of wolves, a bunch of grapes, or a flock of pigeons are all examples of sets of things."

Notation: If **A** is a set, then $\mathbf{x} \in \mathbf{A}$ means that **x** *is an element of* **A**, or equivalently, **x** *is a member of* **A**, or **x** *belongs to* **A** or **x** *is in* **A.** The set with no elements is called the **empty set** and is represented by the symbol **Ø.**

Example: If **C** is the set of positive odd whole numbers less than 10, then $7 \in \mathbf{C}$. In some instances, we can easily describe a set by listing all of its elements within curly braces; thus $\mathbf{D} = \{1,3,5,7,9\}$ also defines a particular set.

Definition: Two sets are **equal** if and only if they have precisely the same elements.

Examples: **1)** The sets **C** and **D** in our first example are equal.

2) The set of women who have served as presidents of the United Sates is equal to **Ø** . Note that this last statement may no longer be true in a few years.

Definitions: If **A** is a set, the statement that **B** is a **subset** of **A** means that **B** is a set, and that each element of **B** is an element of **A.** If **A** is a set the notation $\mathbf{B} \subset \mathbf{A}$ means that **B** is a subset of **A.**

If **A** is a set and **B** is a subset of **A**, then **B** is a **proper subset** of **A** if and only if there is an element of **A** which is not an element of **B.**

Definition: If each of **A** and **B** is a set, the **union** of **A** and **B,** denoted $\mathbf{A} \cup \mathbf{B}$, is the set **C** such that **x** is an element of **C** if and only if either **x** is an element of **A** or of **B. (**Equivalently**, x** is an element of the union of **A** and **B** if and only if **x** is a member of *at least one* of the sets **A** and **B,)**

Definition: If **G** is a collection, each element of which is a set, the **union** of the sets of **G** is the set **X** such that **y** is an element of **X** if and only if there is an element **g** of **G** such that **y** is an element of **g**.

Definition: If each of **A** and **B** is a set, then the **intersection**, or **common part** of **A** and **B,** denoted **A ∩ B,** is the set **C** such that **x** is an element of **C** if and only if **x** is an element of **A** and **x** is an element of **B. (**Equivalently**, x** is an element of the intersection of **A** and **B** if and only if **x** is a member of *both* of the sets **A** and **B.)**

Definition: If **A** and **B** are sets and have no element in common, then **A** and **B** are **disjoint**. This is denoted **A ∩ B** = **Ø.**

Definition: If **G** is a nonempty collection of sets, then the **intersection** or **common part** of the sets of **G** is the set **C** such that **x** is an element of **C** if and only if for each set **g** of **G**, **x** is an element of **g**.

Notation: Suppose that **P** is a "well defined property" that an object may or may not possess. We use the notation **{x**: **x** has property **P}** to denote the set of all objects with property **P**.

For example, **{x**: **x** is a real number and **x** > 0**}** is just the set of positive real numbers, **{x**: **x** is a student at Middlebury College**}** is just the collection of students at this college, and if **A** and **B** are sets, then **{x**: either **x** is in **A** or **x** is in B**}** is just the union of **A** and **B.**

We will assume the following axiom about the positive integers. Feel free to use it whenever appropriate.

Axiom of Mathematical Induction: Every non-empty set of positive integers has a smallest element; that is, if **A** is a set all of whose elements are positive integers and **A** is not the empty set, then there exists an element s in **A** such that $s \leq x$ for all x in **A**.

Definition: The statement that **F** is a **function** means that **F** is a collection of ordered pairs, such that no two of these pairs have the same first term.

Definitions: Suppose that **F** is a function. The **domain** of **F** is the set **X** such that **x** is an element of **X** if and only if **x** is the first term of some element of **F**. The **range** of **F** is the set **Y** such that **y** is an element of **Y** if and only if **y** is a second term of some element of **F**. If **x** is the first term of an element of **F**, then **F**(**x**), the **value of F at x**, denotes the second term of the ordered pair of **F** whose first term is **x** The function **F** is said to be a function from **X onto Y**. If **Z** is a set such that **Y** is a subset of **Z**, then **F** is a function from **X into Z**. The notation **F:X** → **Z** means that **F** is a function from **X** into **Z**. If **A** is a subset of **X,** then **F**(**A)** denotes the set of all elements **F**(a) where a is an element of **A.**

Example 3: Let $\mathbb{R}$ denote the set of all real numbers. Let **F** be $\{(\mathbf{x},\mathbf{x}^2)$: **x** is an element of $\mathbb{R}$ } . Then **F** is a function. The domain of **F** is $\mathbb{R}$, the range of **F** is {**x**: **x** is in $\mathbb{R}$ and $\mathbf{x} \geq 0$}, **F**(2) = 4, and we have $\mathbf{F}:\mathbb{R} \rightarrow \mathbb{R}$

Definition: Suppose that **F** is a function. The statement that **F** is **one-to-one** means that no two elements of **F** have the same second term. In other words, if **x** and **y** are distinct elements of the domain of **F**, then **F**(x) is different from **F**(y). Note in the above example, **F** is not one-to-one because **F**(2) = **F**(-2).

Question 1: Let **f** be a function from a set **X** into a set **Y** and let **A** and **B** be subsets of **X**. Which of the following statements are always true?

(a) $\mathbf{f(A \cup B) = f(A) \cup f(B)}$

(b) $\mathbf{f(A \cap B) = f(A) \cap f(B)}$

(c) **Y - f(A)** = **f(X - A)** for each subset **A** of **X**.

Note that if **C** and **D** are sets, then **C** - **D** denotes the set of all elements of **C** which are not members of **D**.

(d) If **f** is a one-to-one function and **g** is a subset of **f**, then **g** is a one-to-one function.

(e) Does statement (a) remain true if the union of two sets is replaced by the union of an arbitrary collection of subsets of **X**?

(f) Does statement (b) remain true if the intersection of two sets is replaced by the intersection of an arbitrary collection of subsets of **X**?

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Definition: Suppose that **X** and **Y** are sets. Then the statement that **X is equivalent to Y**, denoted **X ~ Y**, means that there is a one-to-one function from **X** onto **Y**.

Example 4: Let **F** = { (*n*, *n*+1): *n* is a non-negative integer}. Then **F** is a one-to-one function from the set of non-negative integers onto the set of positive integers.

Notation: Henceforth, we will let **J** denote the set of positive integers, $\mathbb{R}$ denote the set of real numbers, and **Q** denote the set of rational numbers.

Example 5: Let **F** be { (x, arctan x): x is in $\mathbb{R}$} . Then **F** is a one-to-one function from $\mathbb{R}$ onto {x: x is in $\mathbb{R}$ and $-\pi/2 < x < \pi/2$} .

Example 6: Let **G** be { (x, exp(x)): x is in $\mathbb{R}$} where exp(x) is the usual exponential function. Then **G** is a one-to-one function from $\mathbb{R}$ onto {x: x is in $\mathbb{R}$ and $x > 0$} .

Example 7: Let **H** be { $(x, \frac{1}{1+x^2})$) : x is a positive real number } Then **H** is a one-to-one function from the positive real numbers onto {x: x is in $\mathbb{R}$ and $0 < x > 1$} , the open interval (0,1).

Exercise 1: Show that if **A** is the set of positive integers and **B** is the set of even positive integers, then **A** ~ **B**

Exercise 2: Show that if **A** is the set of positive integers and **Z** is the set of integers, then **A** ~ **Z**.

Exercise 3: Let **C** = {$1/n$: n is a positive integer} and let **D = C** $\cup$ **{0}.** Show that **C** ~ **D**

Exercise 4: Let **A** be the closed unit interval; that is, **A** = {x: x is a real number and $0 \le x \le 1$}. Let **B** be the open unit interval; that is, **B** = {x : x is a real number and $0 < x < 1$ }. Is **A~ B**? Is **B ~ A** ?

Exercise 5: Find a one-to-one function from the open interval (0,1) onto the positive real numbers.

Theorem 1: If **X** is a set, then **X ~ X**.

Theorem 2: Suppose **X** and **Y** are sets. If **X ~ Y**, then **Y ~X**.

Theorem 3: Suppose that **X, Y**, and **Z** are sets. If **X ~ Y** and **Y ~ Z**, then **X ~ Z**.

Definition: Suppose that **A** is a set. Then **A** is **infinite** if and only if **A** is equivalent to a proper subset of itself. A set **A** is **finite** if and only if it is not infinite.

Exercise 6: **J** is infinite

Exercise 7: **Q** is infinite

Exercise 8: $\mathbb{R}$ is infinite

Exercise 9: The empty set is finite.

Exercise 10: The set **A** = {1} is finite.

Exercise 11: Show that the set of positive integers can be written as the union of two disjoint infinite sets.

Exercise 12: Show that the set of positive integers can be written as the union of three pairwise disjoint infinite sets.

Definition: Suppose that **A** is a set. Then **A** is **countable** if and only if there is a subset **H** of **J** such that **A** ~ **H**.

Theorem 4: Suppose **A** and **B** are sets. If **A** ~ **B** and **A** is infinite, then **B** is infinite.

Theorem 5: Suppose that **A** and **B** are sets. If **A** is a subset of **B** and if **A** is infinite, then **B** is infinite.

Theorem 6: Suppose that **A** is a set, **B** is a subset of **A**, **C** is a subset of **B** and **A** ~ **C**. Then **A** ~ **B.**

Theorem 7: Suppose that **A** is a set, **M** is a set, **B** is a subset of **A**, **N** is a subset of **M**, **A** ~ **N** and **M** ~ **B.** Then **A** ~ **M**.

Exercise 13: Show that the set of positive integers can be written as the union of a countable collection of pairwise disjoint infinite sets.

Theorem 8: If **A** is a countable set and **B** is a countable set, then the union **A** $\cup$ **B** is a countable set.

Theorem 9: If **G** is a countable collection of sets, each of which is countable, then the union of the sets of **G** is countable.

Notation: If **A** is a set, we will let $\mathbf{2^A}$ denote the collection of all subsets of **A.**

Question 2: Does there exist a set **A** such that **A** ~ $\mathbf{2^A}$?

Definitions: Suppose **a** and **b** are real numbers and that **a** < **b**. The **closed interval** from **a** to **b,** denoted **[a,b]** is **{x**: **x** is a real number and **a** ≤ **x** and **x** ≤ **b}** while the **open interval** from **a** to **b**, denoted **(a,b)** is the set **{x**: **x** is a real number, **a** < **x** and **x** < **b}** .

We will assume the following axiom about the real numbers. Feel free to use it whenever appropriate.

Axiom: The intersection of any countable nested collection of closed intervals of real numbers is nonempty. By a **nested collection,** we mean a collection in which the $i+1$st interval is a subset of the i th interval for each $i = 1,2,3,...$

***Theorem* 10:** Suppose n is a positive integer and $Z_n = \{z: z$ is an element of $\mathbf{J}$ and $z \leq n\}$. Then Z_n is finite.

***Theorem* 11**: If $m \neq n$, then Z_m is not equivalent to Z_n.

***Theorem* 12:** If **A** is an infinite set and p is an element of **A**,
then **A** ~ **A** – $\{p\}$

***Theorem* 13**: If **A** is an infinite subset of **J**, then **A** ~ **J**.

***Theorem* 14:** If **A** and **B** are infinite sets and **A** ~ **B**, then **A** ~ **A** $\cup$ **B**.

***Theorem* 15:** If **A** and **B** are infinite countable sets, then **A** ~ **B.**

***Theorem* 16:** If **A** and **B** are infinite sets which are not countable,
then **A** ~ **B**.

Theorem **17**: Suppose that **X** is a non-empty finite set. Then there is a positive integer **n** such that **X** is equivalent to **{ z: z** is an element of **J** and **z ≤ n}.**