

Sequences and Series

Sequence	Geometric Progression
Limit of a sequence	Geometric Series
Series	Harmonic Series
Partial Sum	Euler's Constant (Gamma γ)

Definition: A **sequence** is a function whose domain is the set of positive integers.

Notation. If S is a sequence, then $S = \{(n, a_n) : n \in \mathbb{N}\}$ and we often write $\{a_n\}$ as a shorthand notation.

Definition: A **real-valued sequence** or a **sequence of real numbers** is a sequence whose range is a subset of the real numbers; that is, each a_n is a real number.

Definition. A **geometric progression** is a sequence $\{a_n\}$ where there is a real number r such $a_{n+1} = ra_n$ for all positive integers n . The number r is called the **ratio** for the geometric progression.

An **arithmetic sequence** is a sequence $\{a_n\}$ where there is a real number b such $a_{n+1} = a_n + b$ for all positive integers n .

Definition: A sequence of real numbers $\{a_n\}$ **converges** to a number L if for every $\varepsilon > 0$, there is some positive integer N such that $|a_n - L| < \varepsilon$ for every $n > N$

A sequence **diverges** if it does not converge.

Notation: If a sequence $\{a_n\}$ converges to L , then we write $a_n \rightarrow L$ or $\lim_{n \rightarrow \infty} a_n = L$

Example: The sequence $\left\{ \left(\frac{1}{2} \right)^n \right\}$ converges to 0.

Proof: $\left(\frac{1}{2} \right)^n < \varepsilon \Leftrightarrow n \log \left(\frac{1}{2} \right) < \log \varepsilon \Leftrightarrow n > \frac{\log \varepsilon}{\log(\frac{1}{2})}$. Choose any integer

N which is larger than $\frac{\log \varepsilon}{\log(\frac{1}{2})}$

Theorem: If $|r| < 1$, then $\{r^n\}$ converges to 0.

Proof: $|r^n| < \varepsilon \Leftrightarrow |r|^n < n \log |r| < \log \varepsilon \Leftrightarrow n > \frac{\log \varepsilon}{\log(|r|)}$. Choose any

integer N which is larger than $\frac{\log \varepsilon}{\log(|r|)}$.

Definitions: A **constant sequence** $\{c\}$ is a sequence $\{a_n\}$ in which there is a number c such that $a_n = c$ for all positive integers n .

If $\{a_n\}$ and $\{b_n\}$ are sequences then their **sum** is the sequence $\{a_n + b_n\}$, their **difference** is the sequence $\{a_n - b_n\}$, their **product** is $\{a_n b_n\}$ and their **quotient**, if no $b_n = 0$, is $\{a_n / b_n\}$.

Theorem: If $\{a_n\}$ converges to L and $\{b_n\}$ converges to M , then

- (1) $\{c a_n\}$ converges to $c L$ for any real number c .
- (2) $\{a_n + b_n\}$ converges to $L + M$
- (3) $\{a_n - b_n\}$ converges to $L - M$
- (4) $\{a_n b_n\}$ converges to $L M$
- (5) $\{a_n / b_n\}$ converges to L / M if no b_n equals 0 and M is nonzero.

Example: If $|r| < 1$, then the sequence $\left\{ \frac{\alpha(1-r^n)}{1-r} \right\}$ converges to $\frac{\alpha}{1-r}$

Example: If a and r are real numbers with $|r| < 1$, then the sequence $\{S_n\}$ converges to $\frac{\alpha}{1-r}$ if $S_n = a + ar + ar^2 + \dots + ar^{n-1} = \sum_{k=1}^n ar^{k-1}$

Definition: If $\{a_n\}$ is any sequence of real numbers, then we can form a new related sequence $\{S_n\}$ called the **sequence of partial sums** of

$\{a_n\}$ by letting $S_n = a_1 + a_2 + a_3 + \dots + a_n = \sum_{k=1}^n a_k$

Definitions: If $\{a_n\}$ is a sequence of real numbers, then an **infinite series** is a formal expression of the form

$$a_1 + a_2 + a_3 + \dots + a_n + \dots = \sum_{k=1}^{\infty} a_k .$$

We say the **infinite series converges** if there is a real number L so that the sequence of partial sums $\{S_n\}$ converges to L . An infinite series **diverges** if it does not converge.

A **geometric series** is an infinite series where the underlying sequence is a geometric progression. The number $a = a_1$ is called the **first term** of the series and r is called the **ratio**.

Theorem. A geometric series with first term a and common ratio r converges to $\frac{a}{1-r}$ if $|r| < 1$. If $a \neq 0$ and $|r| \geq 1$, then the series diverges.

Theorem. The harmonic series $\sum_{n=1}^{\infty} \frac{1}{n} = 1 + \frac{1}{2} + \frac{1}{3} + \frac{1}{4} + \dots + \frac{1}{n} + \dots$ diverges.