

HOMEWORK #3
(due Wednesday, September 27)

Read: Bertsekas and Tsitsiklis, §§1.3-1.5.

1. The *inclusion-exclusion formula* for three events is

$$P(A \cup B \cup C) = P(A) + P(B) + P(C) - P(AB) - P(AC) - P(BC) + P(ABC)$$

(looking at a Venn diagram shows the intuition).

- (a) Write $A \cup B \cup C = (A \cup B) \cup C$ and derive the above from the two-event version.
- (b) Apply the new formula to find the probability of at least one ace in three rolls of a die.
- (c) Check your result from (b) by using the complement rule to compute the probability.
2. How many people do you need in a room in order to have a better than even chance that at least one pair have the same Zodiac sign? What are you assuming? How many people would you have to interview (in a bar?!) to have a better than even chance of meeting someone with your Zodiac sign?
3. Consider the experiment of rolling a four-sided die (numbered 1,2,3,4) until the first time you observe a number that you have already seen. Let p_r denote the probability that you roll exactly r times.
- (a) Without computation, state the value of $p_1 + p_2 + p_3 + \dots + p_6$. Explain.
- (b) Now find the values of p_r for $r = 1, 2, 3, \dots$
- (c) Check that your values from (b) match your answer from (a)!
4. Over the years, gamblers have developed an impressive array of techniques for cheating. *Bricks* or *flats* are dice that have one side shaved down so the four adjoining sides are made smaller and have less surface area. The shaved side and its opposite face will be the more likely outcome. Six-ace flats (like the die on the right) will favor the 1 and 6 sides and work against the shooter in a game of Craps. What is the chance of winning the pass bet with a pair of six-ace flats engineered so that $p_1 = p_6 = 1/4$ and $p_2 = p_3 = p_4 = p_5 = 1/8$?



Square die and a "brick"

5. A fair die is rolled three times. Let E be the event “the sum of the rolls is 10” and F be the event “the first roll is 4.” Find $P(E|F)$ and $P(F|E)$.
6. Here is a formula for $P(A|B)$ by conditioning on cases of B . If B_1, B_2, \dots, B_n is a partition of B then

$$P(A|B) = P(A|B_1)P(B_1|B) + P(A|B_2)P(B_2|B) + \dots + P(A|B_n)P(B_n|B)$$

- (a) Show that the formula is correct.
- (b) Demonstrate the use of the formula by computing $P(W|D)$ in the game of craps (with ordinary dice), where W is the event that you win the “Pass” bet and D is the event that your first roll is doubles. (Use the partition $D = D_1 \cup D_2 \cup \dots \cup D_6$).
7. A box contains three shaped dice, as described in problem 4. One is a fair die, and therefore lands “flat” (ace or six) with probability $1/3$. The other two shapes land flat with probability $1/2$ and $2/3$ respectively. One of these dice will be chosen at random and rolled.
- (a) What is the chance that a six will be rolled?
- (b) Given that the number rolled is six, what is the chance that the fair die was chosen?
8. *Polya’s Urn Model.* An urn contains 4 white balls and 6 black balls. A ball is drawn at random and its color is noted. The ball is then replaced in the urn, along with 3 more balls of the same color (so there are now 13 balls in the urn). Now another ball is drawn at random.
- (a) Find the probability that the second ball drawn is white.
- (b) Given that the second ball is white, what is the probability the added balls were black?
- (c) Suppose that the urn initially contains w white balls and b black balls, and that when a ball is drawn it is replaced with d balls of the same color (above we had $w = 4$, $b = 6$ and $d=3$). Show that the chance that the second ball drawn is white is $w/(w + b)$.
[Interestingly, this does not depend on the value of d !]