

**HOMEWORK #7**  
**(due Friday, October 27)**

*Read:* Bertsekas and Tsitsiklis, §§2.5-2.8.

1. *Chuck-a-luck*. This game can often be found at carnivals. A player bets \$1 on any one of the numbers 1, 2, 3, 4, 5, 6. Three dice are rolled. If the player's number appears  $k$  times, for any  $k \geq 1$ , then the player gets  $\$k$  back, plus the original stake of \$1. Otherwise, the player loses the \$1 stake. Consider the following argument for playing. Since you have a  $1/6$  chance on each die, you have at least a  $1/6 + 1/6 + 1/6 = 1/2$  chance that your number will come up once, doubling your money. This is already enough to break even, so the possible extra payoff from your number coming up more than once actually makes the game favorable!
  - (a) This can't be right, or they wouldn't offer the game. Find the flaw in this reasoning.
  - (b) Over the long run, how many cents per game should a player expect to win or lose at Chuck-a-luck?
  
2. Recall from class that a Las Vegas roulette wheel is divided into 38 congruent sectors, numbered  $\{0, 00, 1, 2, \dots, 36\}$ . A bet on a single number, say on #13, pays 35:1, so the expected return on a \$1 bet is:  $(+35)(1/38) + (-1)(37/38) = -2/38$ . Suppose that Mr. Dorman compulsively places \$1 bets on #13. To cure this habit, Mr. Peterson offers to bet \$20 that Mr. Dorman will be behind after 36 spins of the roulette wheel (if Mr. Dorman has lost money at roulette, he pays \$20 to Mr. Peterson; if he is ahead, then Mr. Peterson pays him \$20). Show that this bet is actually unfavorable for Mr. Peterson! (*Hint:* there is only one scenario in which Mr. Dorman is behind after 36 spins.) How do you explain this?
  
3. Let the random variable  $N$  be the smaller of the two numbers obtained when rolling a pair of dice. You found the PMF of  $N$  in HW 6.1.
  - (a) Use the PMF to find  $E[N]$ .
  - (b) Now apply the "tail sum formula" for expectation to find  $E[N]$ . *Note:* You can of course find the upper tail probabilities from the PMF. But a slicker way is to observe that the minimum of two numbers exceeds  $x$  if and only if both numbers exceed  $x$ .
  - (c) Find the variance of  $N$ .
  
4. Let  $X$  be a Poisson random variable with parameter  $\lambda$ . Find  $E[1/(1+X)]$ .

5. An urn contains a red ball and a blue ball. A ball is drawn at random. If it is red, then the game is over. If it is blue, then the ball is returned to the urn along with an extra blue ball. The drawing continues until the red ball is drawn, however long this takes. Let  $Y$  be the number of draws required. Show that that  $P(Y = k) = 1/[k(k+1)]$ ,  $k = 1, 2, 3, \dots$ . Show that this defines a valid PMF, but that  $E[Y]$  does not exist.
6. Recall from class the *St. Petersburg Paradox*, which involves a bet paying  $\$2^k$  if your first head appears on the  $k^{\text{th}}$  toss. But can the house really pay off no matter *what* happens? More realistically, suppose that the house has a total of  $2^m$  dollars. Then for any number of tosses larger than  $m$ , you would still receive only  $2^m$  dollars (having "broken the bank").
- (a) Show that the game is now worth only  $m + 1$  dollars in expectation.
- (b) What is the most that you should pay to play against a friend who was no prepared to risk more than \$64? Against Donald Trump on reality television (an oxymoron if ever there were) if he were willing to pay up to \$1,000,000?
7. A fair coin is tossed until the first time that at least one head and at least one tail have been observed. Let  $N$  be the number of tosses required.
- (a) Find  $E[N]$  by relating it to a "brand name" random variable.
- (b) What is the PMF of  $N$ ?
8. Consider the *triangular PMF* defined by: 
$$p_X(k) = \frac{2k}{n(n+1)}, \quad k = 1, 2, \dots, n.$$
- (a) Sketch the PMF for the case  $n = 4$ . Find the mean and variance of this distribution by direct computation.
- (b) Now find a formula for the mean and variance with general  $n$ .
9. Let  $X$  be the number of heads obtained in three tosses of a fair coin.
- (a) Compute the mean absolute deviation  $E(|X - \mu|)$ . Your answer should be slightly smaller than the standard found in class.
- (b) This is a general phenomenon: the operation of squaring the deviations before averaging gives greater weight to the larger (absolute) deviations. Let  $X$  be a random variable with mean  $\mu$  and standard deviation  $\sigma$ . Use the fact that  $\text{Var}(|X - \mu|) \geq 0$  to show that  $\sigma > E(|X - \mu|)$ , with equality only if  $|X - \mu|$  is a constant.