MATH 310 Fall 2006

HOMEWORK #9
(due Friday, November 10)

Read: Bertsekas and Tsitsiklis, §§3.1-3.2.

1. Use the Total Expectation Theorem to find the expected number of times you will roll to settle a "pass" bet in the game of craps.

2. A fair coin is tossed four times. Let X be the number of heads obtained, and Y be the number of tails that appear before the first head.

   (a) Find the marginal PMF for Y. [Observe that $Y = 0$ for outcomes of the form $h\_\_\_$ (where $\_$ can be h or t); $Y = 1$ for outcomes $th\_\_$; … ; and $Y = 4$ for $tttt$.]

   (b) Now construct the joint PMF table by the following steps. For each y, fill in the joint probabilities $P(X = x, Y = y)$ by considering $\mathrm{P}(X = x \mid Y = y)$ and using the marginal PMF of Y from part (a). Of course, the marginal PMF for X is an old friend, and you can use this as a check on your work.

   (c) For each $y = 0,1,2,3,4$, find $E[X \mid Y = y]$. Then apply the Total Expectation Theorem to compute $E[X]$. Again, you can check your work against the known value of $E[X]$.

   (d) Find $\mathrm{Cov}(X, Y)$.

3. Let T be the number of trials until the r^{th} success in a sequence of independent trials with success probability p. Recall that T has the negative binomial PMF

$$p_T(k) = \binom{k-1}{r-1} p^r q^{k-r}, \qquad k = r, r+1, \ldots$$

   You might not enjoy finding $E[T]$ or $E[T^2]$ directly. Instead, express T as a sum of simpler random variables, and then use this representation to find $E[T]$ and $\mathrm{Var}(T)$. Finally, use these values to find $E[T^2]$.

4. Suppose that $E[X] = 3$, $E[X^2] = 14$, $E[Y] = 5$, $E[Y^2] = 31$, and $E[XY] = 13$. Find the following.

   (a) $\mathrm{Var}(X)$ (b) $\mathrm{Cov}(X, Y)$ (c) $\mathrm{Var}(X + Y)$.

5. *Properties of Covariance*. Let X, Y, Z and W be random variables on Ω, and let a, b, c and d be constants.

   (a) Show that $\mathrm{Cov}(aX + b, Y) = a\mathrm{Cov}(X, Y)$. What then is $\mathrm{Cov}(aX + b, cY + d)$?

   (b) Show that $\mathrm{Cov}(X + Y, Z) = \mathrm{Cov}(X, Z) + \mathrm{Cov}(Y, Z)$. What then is $\mathrm{Cov}(X + Y, Z + W)$?

6. Recall from class the example of placing a \$1 bet on Red and a \$1 bet on the 2nd Column, for the same spin of the roulette wheel. Let I_R and I_C be the indicators for winning the respective bets.

   (a) Find $\mathrm{Cov}(I_R, I_C)$.

   (b) We observed that your net winnings on the bets can be expressed as $X = 2I_R - 1$ and $Y = 3I_C - 1$, respectively. Apply (5a) to calculate $\mathrm{Cov}(X, Y)$. [You should match the value of $-96/361$ obtained in class.]

   (c) More generally, consider two events, A and B, and their respective indicators. Find a formula for $\mathrm{Cov}(I_A, I_B)$ in terms of $P(A)$, $P(B)$ and $P(AB)$.

7. Which of the following can be probability density functions? For those that can, find the value of c and the CDF $F(x)$, and sketch the density.

   (a) $$f(x) = \begin{cases} cx(1-x), & 0 \le x \le 1 \\ 0, & \text{otherwise} \end{cases}$$

   (b) $$f(x) = \begin{cases} cx^{-1}, & x \ge 1 \\ 0, & \text{otherwise} \end{cases}$$

   (c) $$f(x) = \begin{cases} cx^{-2}, & x \ge 1 \\ 0, & \text{otherwise} \end{cases}$$

8. Hint for #7(a): this one works! Now let X denote the corresponding random variable, and find $P\{X \le 1/3\}$, $P\{X > 1/2\}$, and $P\{1/3 \le X \le 1/2\}$.

9. Find the means, if they exist, for the valid densities from problem 7. Find the variances, if they exist.